Assignment 7

Hand in no. 2, 8, 9, 10, and 11 by November 1, 2019.

1. Determine whether $\mathbb{Z}$ and $\mathbb{Q}$ are complete sets in $\mathbb{R}$.
2. Does the collection of all differentiable functions on $[a, b]$ form a complete set in $C[a, b]$?
3. Let (X, d) be a metric space and $C_b(X)$ the vector space of all bounded, continuous functions in X. Show that it forms a complete metric space under the sup-norm.
4. We define a metric on $\mathbb{N}$, the set of all natural numbers by setting
$$d(n, m) = \left| \frac{1}{n} - \frac{1}{m} \right| .$$
   (a) Show that it is not a complete metric.
   (b) Describe how to make it complete by adding one new point.
5. Optional. Let (X, d) be a metric space. Fixing a point $p \in X$, for each x define a function
$$f_x(z) = d(z, x) - d(z, p).$$
   (a) Show that each f_x is a bounded, uniformly continuous function in X.
   (b) Show that the map $x \mapsto f_x$ is an isometric embedding of (X, d) to $C_b(X)$. In other words,
$$\|f_x - f_y\|_\infty = d(x, y), \quad \forall x, y \in X.$$
   (c) Deduce from (b) the completion theorem asserting that every metric space has a completion.

   This approach is much shorter than the proof given in the appendix of Chapter 3 . However, it is not so inspiring.
6. Let $f : E \to Y$ be a uniformly continuous map where $E \subset X$ and X, Y are metric spaces. Suppose that Y is complete. Show that there exists a uniformly continuous map F from $\overline{E}$ to Y satisfying $F = f$ in E. In other words, f can be extended to the closure of E preserving uniform continuity.
7. Consider maps from $\mathbb{R}$ to itself. Provide explicit examples of continuous maps with exactly one, two and three fixed points, and one map satisfying $|f(x) - f(y)| < |x - y|$ but no fixed points.
8. Let T be a continuous map on the complete metric space X. Suppose that for some k, T^k becomes a contraction. Show that T admits a unique fixed point. This generalizes the contraction mapping principle in the case $k = 1$.
9. Show that the equation $2x \sin x - x^4 + x = 0.001$ has a root near $x = 0$.
10. Can you solve the system of equations
$$x + y^4 = 0, \quad y - x^2 = 0.015 \ ?$$
11. Can you solve the system of equations
$$x + y - x^2 = 0, \quad x - y + xy \sin y = -0.005 \ ?$$